Annals of Fuzzy Mathematics and Informatics
Volume 1, No. 2, (April 2011), pp. 171- 188
ISSN 2093–9310
http://www.afmi.or.kr

Right h-weakly regular hemirings

Muhammad Shabir, Rukhshanda Anjum

Received 9 September 2010; Accepted 8 December 2010

Abstract. In this paper we characterize those hemirings for which each right h-ideal is idempotent. We also characterize those hemirings for which each right fuzzy h-ideal is idempotent. We have given the concept of right pure h-ideals, purely prime h-ideals, fuzzy right pure h-ideals and fuzzy purely prime h-ideals and characterize hemirings by these ideals.

2010 AMS Classification: 03G10, 03G25, 08A2

Keywords: Hemirings, h-Right weakly regular, Fuzzy prime right ideals, Pure h-ideals, Fuzzy pure h-ideals.

Corresponding Author: Muhammad Shabir (mshabirbhatti@yahoo.co.uk)

1. Introduction

The notion of semiring was introduced by H. S. Vandiver in 1934 [17]. Semirings provide a common generalization of rings and distributive lattices, appear in a natural manner in some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics (see for example [1, 5]). Hemirings, semirings with commutative addition and zero element, have also proved to be an important algebraic tool in theoretical computer science (see for instance [7]).

The theory of fuzzy sets, proposed by Zadeh [20], has provided a useful mathematical tool for describing the behavior of systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields as artificial intelligence, computer science, control engineering, expert systems, management science, robotics and others. The notions of automata and formal languages have been generalized and extensively studied in a fuzzy frame work (cf. [13, 16, 18]).

Ideals play an important role in the structure theory of hemirings and are useful for many purposes but they do not coincide with usual ring ideals. For this reason many results in ring theory have no analogues in hemirings using only ideals. Henriksen defined in [8] a class of ideals in semirings, which is called the class of k-ideals.

A more restricted class of ideals namely h-ideals has been given by Iizuka [9]. The properties of h-ideals and k-ideals of hemirings were thoroughly investigated by La Torre [12] and by using h-ideals and k-ideals La Torre established some analogous ring theorems for hemirings.

Investigations of fuzzy semirings were initiated in [2]. Fuzzy k-ideals are studied in [6, 11]. Fuzzy h-ideals of a hemiring are studied by many authors for example [4, 10, 14, 15, 19, 21, 22, 23]. In this paper we characterize those hemirings in which each right h-ideal is idempotent. We also characterize those hemirings in which each fuzzy right h-ideal is idempotent. We also study the right pure h-ideals, purely prime h-ideals and their fuzzy version in hemirings.

2. Preliminaries

A *semiring* is a non-empty set R together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ such that $(R,+)$ and $(R,\cdot)$ are semigroups and the following distributive laws:

$$a\cdot(b+c)=a\cdot b+a\cdot c \text{ and } (b+c)\cdot a=b\cdot a+c\cdot a$$

hold for all $a,b,c\in R$.

An *absorbing zero element* means an element $0\in R$ such that $a+0=0+a=a$ and $a\cdot 0=0\cdot a=0$ for all $a\in R$.

A semiring $(R,+,\cdot)$ is called a *hemiring* if $(R,+)$ is a commutative semigroup with an absorbing *zero.*

By the *identity* of a hemiring $(R,+,\cdot)$ we mean an element $1\in R$ (if it exists) such that $1\cdot a=a\cdot 1=a$ for all $a\in R$.

A hemiring $(R,+,\cdot)$ is called commutative if $(R,\cdot)$ is *commutative semigroup.*

A non-empty subset I of a hemiring R is called a *left (right) ideal* of R if

(i) $a+b\in I$ and (ii) $ra\in I$ $(ar\in I)$ for all $a,b\in I$, $r\in R$.

Obviously $0\in I$ for any left (right) ideal I of R.

A non-empty subset A of a hemiring R is called an *ideal* of R if it is both a left and a right ideal of R.

A left (right) ideal A of a hemiring R is called a *left* (*right*) *k-ideal* of R if for any $a,b\in A$ and $x\in R$ from $x+a=b$ it follows $x\in A$.

A left (right) ideal I of a hemiring R is called a *left (right) h-ideal* of R if for any $a,b\in I$ and $x,y\in R$ from $x+a+y=b+y$ it follows $x\in I$. Every left (right) h-ideal is a left (respectively, right) k-ideal. The converse is not true [10].

The h-closure of a non-empty subset A of a hemiring R is defined as

$$\overline{A}=\{x\in R\mid x+a+y=b+y \text{ for some } a,b\in A,\ y\in R\}.$$

It is clear that if A is a left (right) ideal of R, then $\overline{A}$ is the smallest left (right) h-ideal of R containing A. Also, $\overline{A}=A$ for all left (right) h-ideals A of R. Obviously $\overline{\overline{A}}=\overline{A}$ for each non-empty $A\subseteq R$. Also $\overline{A}\subseteq\overline{B}$ for all $A\subseteq B\subseteq R$.

Lemma 2.1 ([23]). $\overline{AB}=\overline{\overline{A}\,\overline{B}}$ *for any subsets* A, B *of a hemiring* R.

Lemma 2.2 ([23]). *If A and B are, respectively, right and left h-ideals of a hemiring R, then*

$$\overline{AB}\subseteq A\cap B.$$

Definition 2.3 ([23])**.** A hemiring R is said to be h-hemiregular if for each $a \in R$, there exist $x, y, z \in R$ such that $a + axa + z = aya + z$.

Theorem 2.4 ([23])**.** *A hemiring R is h-hemiregular if and only if for any fuzzy right h-ideal A and any fuzzy left h-ideal B, we have*

$$\overline{AB} = A \cap B.$$

Let X be a non-empty set. By a *fuzzy subset* μ of X we mean a membership function $\mu : X \to [0,1]$. A fuzzy subset $\mu : X \to [0,1]$ is non-empty if there exist at least one $x \in X$ such that $\mu(x) > 0$. For any fuzzy subsets λ and μ of X we define

$$\begin{aligned} &\lambda \leq \mu \iff \lambda(x) \leq \mu(x), \\ &(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) = \min\{\lambda(x), \mu(x)\}, \\ &(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x) = \max\{\lambda(x), \mu(x)\} \end{aligned}$$

for all $x \in X$.

More generally, if $\{\lambda_i : i \in I\}$ is a collection of fuzzy subsets of X, then by the *intersection* and the *union* of this collection we mean the fuzzy subsets

$$\begin{aligned} \left(\bigwedge_{i\in I} \lambda_i\right)(x) &= \bigwedge_{i\in I} \lambda_i(x) = \inf_{i\in I}\{\lambda_i(x)\}, \\ \left(\bigvee_{i\in I} \lambda_i\right)(x) &= \bigvee_{i\in I} \lambda_i(x) = \sup_{i\in I}\{\lambda_i(x)\}, \end{aligned}$$

respectively.

A fuzzy subset λ of a semiring R is called a *fuzzy left (right) ideal* of R if for all $a, b \in R$ we have

(1) $\lambda(a+b) \geq \lambda(a) \wedge \lambda(b)$,
(2) $\lambda(ab) \geq \lambda(b)$, $(\lambda(ab) \geq \lambda(a))$.

Note that $\lambda(0) \geq \lambda(x)$ for all $x \in R$.

Definition 2.5 ([10])**.** A fuzzy left (right) ideal λ of a hemiring R is called a *fuzzy left (right) k-ideal* if $x + y = z \longrightarrow \lambda(x) \geq \lambda(y) \wedge \lambda(z)$ holds for all $x, y, z \in R$ and fuzzy left (right) *h-ideal* if $x + a + y = b + y \longrightarrow \lambda(x) \geq \lambda(a) \wedge \lambda(b)$ holds for all $a, b, x, y \in R$.

Definition 2.6. Let λ be a fuzzy subset of a universe X and $t \in [0,1]$. Then the subset $U(\lambda; t) = \{x \in X \mid \lambda(x) \geq t\}$ is called level subset of λ.

Lemma 2.7. *A fuzzy set λ of a hemiring R is a fuzzy left (right) h-ideal of R if and only if each non-empty level subset $U(\lambda; t)$ is a left (right) h-ideal of R.*

Proposition 2.8 ([10])**.** *Let A be a non-empty subset of a hemiring R. Then a fuzzy set λ_A defined by*

$$\lambda_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases}$$

where $0 \leq s < t \leq 1$, is a fuzzy right (left) h-ideal of R if and only if A is a right (left) h-ideal of R.

Corollary 2.9. *Let A be a non-empty subset of a hemiring R. Then the characteristic function χ_A of A is a fuzzy right h-ideal of R if and only if A is a right h-ideal of R.*

Lemma 2.10 ([10]). *Let μ and ν be fuzzy left (right) h-ideals of R, then $\mu \wedge \nu$ is also a fuzzy left (right) h-ideal of R.*

Proposition 2.11 ([4]). *If A, B are subsets of a hemiring R such that $Im\lambda_A = Im\lambda_B$ then*

(1) $A \subseteq B \longleftrightarrow \lambda_A \leq \lambda_B$,
(2) $\lambda_A \wedge \lambda_B = \lambda_{A\cap B}$.

Definition 2.12 ([19]). The *h-intrinsic product* of two fuzzy subsets μ and ν of R is defined by

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \nu(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a'_j) \wedge \nu(b'_j)\right] \right]$$

and $(\mu \odot_h \nu)(x) = 0$ if x cannot be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$.

Proposition 2.13 ([19]). *Let μ, ν, ω, λ be fuzzy subsets on R. Then*

(1) *$\mu \leq \omega$ and $\nu \leq \lambda \longrightarrow \mu \odot_h \nu \leq \omega \odot_h \lambda$.*
(2) *$\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$ for characteristic functions χ_A and χ_B of any subsets A, B of R.*

Theorem 2.14 ([4]). *(i) If λ and μ are fuzzy h-ideals of R, then so is $\lambda \odot_h \mu$. Moreover, $\lambda \odot_h \mu \leq \lambda \wedge \mu$.*

(ii) If λ and μ are fuzzy right h-ideal and fuzzy left h-ideal of R, then $\lambda \odot_h \mu \leq \lambda \wedge \mu$.

Theorem 2.15 ([19]). *A hemiring R is h-hemiregular if and only if for any fuzzy right h-ideal λ and any fuzzy left h-ideal μ of R we have $\lambda \odot_h \mu = \lambda \wedge \mu$.*

3. Right h-weakly regular hemirings

In this section we define right h-weakly regular hemirings and characterize these hemirings by the properties of their right h-ideals and fuzzy right h-ideals.

Definition 3.1. A hemiring R is called right (left) h-weakly regular hemiring if for each $x \in R$, $x \in \overline{(xR)^2}$ $\left(\text{resp. } x \in \overline{(Rx)^2}\right)$.

That is for each $x \in R$ we have $r_i, s_i, t_j, p_j, z \in R$ such that

$$x + \sum_{i=1}^{n} x r_i x s_i + z = \sum_{j=1}^{n} x t_j x p_j + z$$

$$\left(\text{resp. } x + \sum_{i=1}^{n} r_i x s_i x + z = \sum_{j=1}^{m} t_j x p_j x + z\right).$$

Thus each h-hemiregular hemiring with identity is right h-weakly regular but the converse is not true. However for a commutative hemiring both the concepts coincide.

Proposition 3.2. *The following statements are equivalent for a hemiring R with identity :*

(1) *R is right h-weakly regular hemiring.*
(2) *All right h-ideals of R are h-idempotent (A right h-ideal B of R is h-idempotent if $\overline{B^2} = B$).*
(3) *$\overline{BA} = B \cap A$ for all right h-ideals B and two-sided h-ideals A of R.*

Proof. (1) ⇒ (2) Let R be a right h-weakly regular hemiring and B be a right h-ideal of R. Clearly $\overline{B^2} \subseteq B$.

Let $x \in B$. Since R is right h-weakly regular, so $x \in \overline{(xR)^2}$ where xR is the right ideal of R generated by x and so $\overline{xR}$ is the right h-ideal of R generated by x. Thus $xR \subseteq B$, this implies

$$x \in \overline{(xR)\,(xR)} \subseteq \overline{BB} = \overline{B^2}.$$

Thus

$$B \subseteq \overline{B^2}.$$

So, $\overline{B^2} = B$.

(2) ⇒ (3) Let B be a right h-ideal of R and A be a two-sided h-ideal of R then by Lemma 2.2, $\overline{BA} \subseteq B \cap A$.

To prove the reverse inclusion, let $x \in B \cap A$ and xR and RxR are the right ideal and two-sided ideal of R generated by x, respectively. Thus $xR \subseteq B$ and $RxR \subseteq A$. Now

$$x \in xR \subseteq \overline{xR} = \overline{\overline{xR}\;\overline{xR}} = \overline{xRxR} = \overline{(xR)\,(xR)} = \overline{x\,(RxR)} \subseteq \overline{xA} \subseteq \overline{BA}.$$

Hence $B \cap A \subseteq \overline{BA}$ and so $B \cap A = \overline{BA}$.

(3) ⇒ (1) Let $x \in R$ and RxR and xR be the two-sided ideal and right ideal of R generated by x, repectively. Then

$$x \in xR \cap RxR \subseteq \overline{xR} \cap \overline{RxR} = \overline{\overline{xR}\;\overline{RxR}} = \overline{xRRxR} = \overline{xR^2xR} = \overline{(xR)^2}.$$

Hence R is right h-weakly regular hemiring. □

Theorem 3.3. *The following assertions are equivalent for a hemiring R with identity:*

(1) *R is right h-weakly regular hemiring.*
(2) *All fuzzy right h-ideals of R are h- idempotent (A fuzzy right h-ideal λ of R is idempotent if $\lambda \odot_h \lambda = \lambda$).*
(3) *$\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.*

Proof. (1) ⇒ (2) Let λ be a fuzzy right h-ideal of R, then we have $\lambda \odot_h \lambda \leq \lambda$.

For the reverse inclusion, let $x \in R$. Since R is right h-weakly regular so there exist $s_i, t_i, s_j^{'}, t_j^{'}, z \in R$ such that

$$x + \sum_{i=1}^{m} x s_i x t_i + z = \sum_{j=1}^{n} x s_j^{'} x t_j^{'} + z.$$

Hence

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{i=1}^{m} \left(\lambda(xs_i) \wedge \lambda(xt_i)\right).$$

Also

$$\lambda(x) = \lambda(x) \wedge \lambda(x) \leq \bigwedge_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right).$$

Therefore

$$\begin{aligned}
\lambda(x) &\leq \left(\bigwedge_{i=1}^{m} \left(\lambda(xs_i) \wedge \lambda(xt_i)\right)\right) \wedge \left(\bigwedge_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right)\right) \\
&\leq \bigvee_{x+\sum\limits_{i=1}^{m} xs_ixt_i+z=\sum\limits_{j=1}^{n} xs_j^{'}xt_j^{'}+z} \left[\begin{array}{c} \left(\bigwedge\limits_{i=1}^{m} \left(\lambda(xs_i) \wedge \lambda(xt_i)\right)\right) \\ \wedge \left(\bigwedge\limits_{j=1}^{n} \left(\lambda(xs_j^{'}) \wedge \lambda(xt_j^{'})\right)\right) \end{array}\right] \\
&= (\lambda \odot_h \lambda)(x).
\end{aligned}$$

Hence $\lambda \leq \lambda \odot_h \lambda$. Thus $\lambda \odot_h \lambda = \lambda$.

(2) $\Rightarrow$ (3) Let λ and μ be fuzzy right and two sided h-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right h-ideal of R. By Theorem 2.14, $\lambda \odot_h \mu \leq \lambda \wedge \mu$. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$$

Hence $\lambda \odot_h \mu = \lambda \wedge \mu$.

(3) $\Rightarrow$ (1) Let B be a right h-ideal of R and A be a two-sided h-ideal of R, then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided h-ideals of R, respectively. Hence by hypothesis and Propositions 2.11, and 2.13, we have

$$\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A \implies \chi_{\overline{BA}} = \chi_{B\cap A} \implies \overline{BA} = B \cap A.$$

Thus by Proposition 3.2, R is right h-weakly regular hemiring. 3.3. $\square$

Theorem 3.4. *The following assertions are equivalent for a hemiring R with identity:*

1. *R is right h-weakly regular hemiring.*
2. *All right h-ideals of R are h-idempotent.*
3. *$\overline{BA} = B \cap A$ for all right h-ideals B and two-sided h-ideals A of R.*
4. *All fuzzy right h-ideals of R are h-idempotent.*
5. *$\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.*

If R is assumed to be commutative, then the above assertions are equivalent to

6. *R is h-hemiregular.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 3.2.

(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by Theorem 3.3.

Finally If R is commutative, then by Theorem 2.15, (1) $\Leftrightarrow$ (6). $\square$

Definition 3.5 ([4]). The *h-sum* $\lambda +_h \mu$ of fuzzy subsets λ and μ of R is defined by

$$(\lambda +_h \mu)(x) = \sup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \Big(\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)\Big),$$

where $x, a_1, b_1, a_2, b_2, z \in R$.

Theorem 3.6 ([4]). *The h-sum of fuzzy h-ideals of R is also a fuzzy h-ideal of R.*

Theorem 3.7. *If R is right h-weakly regular hemiring, then the collection of all h-ideals of R forms a complete Brouwerian lattice.*

Proof. The collection $\mathcal{L}_R$ of all h-ideals of right h-weakly regular hemiring R is a poset under the inclusion of sets. It is not difficult to see that $\mathcal{L}_R$ is a complete lattice under the operations $\sqcup$, $\sqcap$ defined as $A \sqcup B = \overline{A+B}$ and $A \sqcap B = A \cap B$.

We show that $\mathcal{L}_R$ is a Brouwerian lattice, that is, for any $A, B \in \mathcal{L}_R$, the set $\mathcal{L}_R(A, B) = \{I \in \mathcal{L}_R \,|\, A \cap I \subseteq B\}$ contains a greatest element.

By Zorn's Lemma the set $\mathcal{L}_R(A, B)$ contains a maximal element M. Since R is right h-weakly regular hemiring, so $\overline{AI} = A \cap I \subseteq B$ and $\overline{AM} = A \cap M \subseteq B$. Thus $\overline{AI} + \overline{AM} \subseteq B$. Consequently, $\overline{\overline{AI} + \overline{AM}} \subseteq \overline{B} = B$.

Since $\overline{I+M} = I \sqcup M \in \mathcal{L}_R$, for every $x \in \overline{I+M}$ there exist $i_1, i_2 \in I$, $m_1, m_2 \in M$ and $z \in R$ such that $x + i_1 + m_1 + z = i_2 + m_2 + z$. Thus

$$dx + di_1 + dm_1 + dz = di_2 + dm_2 + dz$$

for any $d \in D \in \mathcal{L}_R$. As $di_1, di_2 \in DI$, $dm_1, dm_2 \in DM$, $dz \in R$, we have $dx \in \overline{DI + DM}$, which implies $D\left(\overline{I+M}\right) \subseteq \overline{DI+DM} \subseteq \overline{\overline{DI} + \overline{DM}} \subseteq B$. Hence $\overline{D\left(\overline{I+M}\right)} \subseteq B$. This means that $D \cap \left(\overline{I+M}\right) = \overline{D\left(\overline{I+M}\right)} \subseteq B$, i.e., $\overline{I+M} \in \mathcal{L}_R(A, B)$, whence $\overline{I+M} = M$ because M is maximal in $\mathcal{L}_R(A, B)$. Therefore $I \subseteq \overline{I} \subseteq \overline{I+M} = M$ for every $I \in \mathcal{L}_R(A, B)$. □

Corollary 3.8. *If R is right h-weakly regular hemiring, then the lattice $\mathcal{L}_R$ is distributive.*

Proof. Each complete Brouwerian lattice is distributive (cf. [3], 11.11). □

Example 3.9. Consider the hemiring $R = \{0, a, b\}$ with the following operations

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

The ideals of R are $\{0\}, \{0, a\}, \{0, b\}$ and R. Only R itself is an h-ideal of R. The collection of h-ideals is a distributive lattice. $\{0, a, b\}$ is h-idempotent and R is right h-weakly regular hemiring.

Theorem 3.10. *If R is right h-weakly regular hemiring, then the set $\mathfrak{F}_R$ of all fuzzy h-ideals of R (ordered by $\leq$) is a distributive lattice.*

Proof. The set $\mathfrak{F}_R$ of all fuzzy h-ideals of R (ordered by $\leq$) is clearly a lattice under the h-sum and intersection of fuzzy h-ideals. Now we show that $\mathfrak{F}_R$ is a distributive lattice, that is for any fuzzy h-ideals λ, μ, δ of R we have $(\lambda \wedge \delta) + \mu = (\lambda + \mu) \wedge (\delta + \mu)$.

For any $x \in R$,

$$\begin{aligned}
[(\lambda \wedge \delta) + \mu](x) &= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} (\lambda \wedge \delta)(a_1) \wedge (\lambda \wedge \delta)(a_2) \wedge \\ (\mu)(b_1) \wedge (\mu)(b_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} \lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \\ \mu(b_2) \wedge \delta(a_1) \wedge \delta(a_2) \end{bmatrix} \\
&= \bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} \begin{bmatrix} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \wedge \\ [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \end{bmatrix} \\
&= \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\lambda(a_1) \wedge \lambda(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&\quad \wedge \left(\bigvee_{x+(a_1+b_1)+z=(a_2+b_2)+z} [\delta(a_1) \wedge \delta(a_2) \wedge \mu(b_1) \wedge \mu(b_2)] \right) \\
&= (\lambda + \mu)(x) \wedge (\delta + \mu)(x) \\
&= [(\lambda + \mu) \wedge (\delta + \mu)](x).
\end{aligned}$$

□

4. Prime and Fuzzy prime right h-ideals

Definition 4.1. A right h-ideal P of a hemiring R is called h-prime (h-semiprime) right h-ideal of R if for any right h-ideals A, B of R,

$$AB \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P \ \left(A^2 \subseteq P \Rightarrow A \subseteq P\right).$$

P is called an h-irreducible (h-strongly irreducible) right h-ideal of R if for any right h-ideals A, B of R

$$A \cap B = P \Rightarrow A = P \text{ or } B = P \ \left(A \cap B \subseteq P \Rightarrow A \subseteq P \text{ or } B \subseteq P\right).$$

A fuzzy right h-ideal μ of a hemiring R is called a fuzzy h-prime (h-semiprime) right h-ideal of R if for any fuzzy right h-ideals λ, δ of R,

$$\lambda \odot_h \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu \ \left(\lambda \odot_h \lambda \leq \mu \Rightarrow \lambda \leq \mu\right).$$

μ is called a fuzzy h-irreducible (h-strongly irreducible) if for any fuzzy right h-ideals λ, δ of R,

$$\lambda \wedge \delta = \mu \Rightarrow \lambda = \mu \text{ or } \delta = \mu \ \left(\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu \text{ or } \delta \leq \mu\right).$$

Lemma 4.2. (a) *Every h-prime right h-ideal (fuzzy h-prime right h-ideal) of a hemiring R is an h-semiprime right h-ideal (fuzzy h-semiprime right h-ideal) of R.*

(b) *The intersection of h-prime right h-ideal (fuzzy h-prime right h-ideal) of R is an h-prime right h-ideal (fuzzy h-prime right h-ideal)of R.*

Proof. Straightforward. □

Theorem 4.3. *Let R be a right h-weakly regular hemiring. Then each proper right h-ideal of R is the intersection of right h-irreducible h-ideals of R which contain it.*

Proof. Let I be a proper right h-ideal of R and let $\{I_\alpha : \alpha \in \Lambda\}$ be a family of right h-irreducible h-ideals of R which contain I. Clearly $I \subseteq \cap_{\alpha\in\Lambda} I_\alpha$. Suppose $a \notin I$. Then by Zorn's Lemma there exists a right h-ideal I_β such that I_β is maximal with respect to the property $I \subseteq I_\beta$ and $a \notin I_\beta$. We will show that I_β is h-irreducible. Let A, B be right h-ideals of R such that $I_\beta = B \cap A$. Suppose $I_\beta \subset B$ and $I_\beta \subset A$. Then by the maximality of I_β, we have $a \in B$ and $a \in A$. But this implies $a \in B \cap A = I_\beta$, which is a contradiction. Hence either $I_\beta = B$ or $I_\beta = A$. So there exists an h-irreducible h-ideal I_β such that $a \notin I_\beta$ and $I \subseteq I_\beta$. Hence $\cap I_\alpha \subseteq I$. Thus $I = \cap I_\alpha$. □

Proposition 4.4. *Let R be a right h-weakly regular hemiring. If λ is a fuzzy right h-ideal of R with $\lambda(a) = \alpha$, where a is any element of R and $\alpha \in [0,1]$, then there exists a fuzzy h-irreducible right h-ideal δ of R such that $\lambda \leq \delta$ and $\delta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu \text{ is a fuzzy right } h\text{-ideal of } R,\ \mu(a) = \alpha \text{ and } \lambda \leq \mu\}$. Then $X \neq \phi$, since $\lambda \in X$. Let $\mathcal{F}$ be a totally ordered subset of X, say $\mathcal{F} = \{\lambda_i : i \in I\}$. We claim that $\bigvee_{i\in I} \lambda_i$ is a fuzzy right h-ideal of R. For any $x, r \in R$, we have

$$\left(\bigvee_i \lambda_i\right)(x) = \bigvee_i (\lambda_i(x)) \leq \bigvee_i (\lambda_i(xr)) = \left(\bigvee_i \lambda_i\right)(xr)$$

Let $x, y \in R$ and consider

$$\begin{aligned}
&\left(\bigvee_i \lambda_i\right)(x) \wedge \left(\bigvee_i \lambda_i\right)(y) = \left(\bigvee_i (\lambda_i(x))\right) \wedge \left(\bigvee_j (\lambda_j(y))\right)\\
&= \bigvee_j \left[\bigvee_i (\lambda_i(x)) \wedge \lambda_j(y)\right] = \bigvee_j \left[\bigvee_i (\lambda_i(x) \wedge \lambda_j(y))\right]\\
&\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j(x) \wedge \lambda_i^j(y)\right)\right] \leq \bigvee_j \left[\bigvee_i \left[\lambda_i^j(x+y)\right]\right]\\
&= \bigvee_{i,j} \left[\lambda_i^j(x+y)\right] \leq \bigvee_i [\lambda_i(x+y)] = \left(\bigvee_i \lambda_i\right)(x+y)
\end{aligned}$$

where $\lambda_i^j = \max\{\lambda_i, \lambda_j\}$ and note that $\lambda_i^j \in \{\lambda_i : i \in I\}$.

Now, let $x + a + z = b + z$ where $a, b, z \in R$. Then

$$\begin{aligned}
&\left(\bigvee_i \lambda_i\right)(a) \wedge \left(\bigvee_i \lambda_i\right)(b) = \left(\bigvee_i (\lambda_i(a))\right) \wedge \left(\bigvee_j (\lambda_j(b))\right)\\
&= \bigvee_j \left[\left(\bigvee_i (\lambda_i(a))\right) \wedge \lambda_j(b)\right] = \bigvee_j \left[\bigvee_i (\lambda_i(a) \wedge \lambda_j(b))\right]
\end{aligned}$$

$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j (a) \wedge \lambda_i^j (b) \right) \right]$$

$$\leq \bigvee_j \left[\bigvee_i \left(\lambda_i^j (x) \right) \right] \text{ because } \lambda_i^j \text{ is a fuzzy right } h\text{-ideal}$$

$$= \bigvee_{i,j} \left[\lambda_i^j (x) \right] \leq \bigvee_i \left[\lambda_i (x) \right] = \left(\bigvee_i \lambda_i \right) (x)$$

where $\lambda_i^j = \max \{\lambda_i, \lambda_j\}$ and note that $\lambda_i^j \in \{\lambda_i : i \in I\}$.

Thus $\bigvee_i \lambda_i$ is a fuzzy right h-ideal of R. Clearly $\lambda \leq \bigvee_i \lambda_i$ and $\bigvee_i \lambda_i (a) = \bigvee_i (\lambda_i (a)) = \alpha$. Thus $\bigvee_i \lambda_i$ is the l.u.b of $\mathcal{F}$. Hence by Zorn's lemma there exists a fuzzy right h-ideal δ of R which is maximal with respect to the property that $\lambda \leq \delta$ and $\delta (a) = \alpha$.

We will show that δ is fuzzy h-irreducible right h-ideal of R. Let $\delta = \delta_1 \wedge \delta_2$, where δ_1, δ_2 are fuzzy right h-ideals of R. Thus $\delta \leq \delta_1$ and $\delta \leq \delta_2$. We claim that either $\delta = \delta_1$ or $\delta = \delta_2$. Suppose $\delta \neq \delta_1$ and $\delta \neq \delta_2$. Since δ is maximal with respect to the property that $\delta (a) = \alpha$ and since $\delta \lneqq \delta_1$ and $\delta \lneqq \delta_2$, so $\delta_1 (a) \neq \alpha$ and $\delta_2 (a) \neq \alpha$. Hence $\alpha = \delta (a) = (\delta_1 \wedge \delta_2) (a) = (\delta_1) (a) \wedge (\delta_2) (a) \neq \alpha$, which is impossible. Hence $\delta = \delta_1$ or $\delta = \delta_2$. Thus δ is fuzzy h-irreducible right h-ideal of R. □

Theorem 4.5. *Every fuzzy right h-ideal of a hemiring R is the intersection of all fuzzy h-irreducible right h-ideals of R which contain it.*

Proof. Let λ be the fuzzy right h-ideal of R and let $\{\lambda_\alpha : \alpha \in \Lambda\}$ be the family of all fuzzy h-irreducible right h-ideals of R which contain λ. Obviously $\lambda \leq \bigwedge_{\alpha \in \Lambda} \lambda_\alpha$. We now show that $\bigwedge_{\alpha \in \Lambda} \lambda_\alpha \leq \lambda$. Let a be any element of R, then by Proposition 4.4, there exists a fuzzy h-irreducible right h-ideal λ_β such that $\lambda \leq \lambda_\beta$ and $\lambda (a) = \lambda_\beta (a)$. Hence $\lambda_\beta \in \{\lambda_\alpha : \alpha \in \Lambda\}$ and so $\bigwedge_{\alpha \in \Lambda} \lambda_\alpha \leq \lambda_\beta$. Thus $\bigwedge_{\alpha \in \Lambda} \lambda_\alpha (a) \leq \lambda_\beta (a) = \lambda (a)$ $\Rightarrow \bigwedge_{\alpha \in \Lambda} \lambda_\alpha \leq \lambda$. Hence $\bigwedge_{\alpha \in \Lambda} \lambda_\alpha = \lambda$. □

Theorem 4.6. *The following assertions for a hemiring R are equivalent:*

(1) *R is right h-weakly regular hemiring.*
(2) *Each right h-ideal of R is h-semiprime right h-ideal of R.*

Proof. (1) $\Rightarrow$ (2) Suppose R is right h-weakly regular hemiring. Let I, J be right h-ideals of R, such that $J^2 \subseteq I \Rightarrow \overline{J^2} \subseteq I$. By Theorem 3.4, $J = \overline{J^2}$, so $J \subseteq I$. Hence I is an h-semiprime right h-ideal of R.

(2) $\Rightarrow$ (1) Assume that each right h-ideal of R is h-semiprime. Let I be a right h-ideal of R. Then $\overline{I^2}$ is also a right h-ideal of R. Also $I^2 \subseteq \overline{I^2}$. Hence by hypothesis $I \subseteq \overline{I^2}$. But $\overline{I^2} \subseteq I$ always. Hence $I = \overline{I^2}$. Thus by Theorem 3.4, R is right h-weakly regular hemiring. □

Theorem 4.7. *The following assertions for a hemiring R with identity are equivalent:*

(1) *R is right h-weakly regular hemiring.*
(2) *All fuzzy right h-ideals of R are h- idempotent (A fuzzy right h-ideal λ of R is idempotent if $\lambda \odot_h \lambda = \lambda$)*
(3) *$\lambda \odot_h \mu = \lambda \wedge \mu$ for all fuzzy right h-ideals λ and all fuzzy two-sided h-ideals μ of R.*
(4) *Each fuzzy right h-ideal of R is a fuzzy h-semiprime right h-ideal of R.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Theorem 3.3.

(2) $\Rightarrow$ (4) Let λ, δ be any fuzzy right h-ideals of R such that $\lambda \odot_h \lambda \leq \delta$. By (2) $\lambda \odot_h \lambda = \lambda$, so $\lambda \leq \delta$. Thus δ is a fuzzy h-semiprime right h-ideal of R.

(4) $\Rightarrow$ (2) Let δ be any fuzzy right h-ideal of R, then $\delta \odot_h \delta$ is also a fuzzy right h-ideal of R and so by (4) $\delta \odot_h \delta$ is a fuzzy h-semiprime right h-ideal of R. As $\delta \odot_h \delta \leq \delta \odot_h \delta \Rightarrow \delta \leq \delta \odot_h \delta$ but $\delta \odot_h \delta \leq \delta$ always. So $\delta \odot_h \delta = \delta$. □

Theorem 4.8. *If every right h-ideal of a hemiring R is h-prime right h-ideal then R is right h-weakly regular hemiring and the set of h-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each right h-ideal is prime right h-ideal. Let A be a right h-ideal of R then $\overline{A^2}$ is a right h-ideal of R. As $A^2 \subseteq \overline{A^2} \Longrightarrow A \subseteq \overline{A^2}$. But $\overline{A^2} \subseteq A$ always. Hence $A = \overline{A^2}$. Thus R is right h-weakly regular hemiring.

Let A, B be any h-ideals of R then $AB \subseteq A \cap B$. As $A \cap B$ is an h-ideal of R, so an h-prime right h-ideal. Thus either $A \subseteq A \cap B$ or $B \subseteq A \cap B$. That is, either $A \subseteq B$ or $B \subseteq A$. □

Theorem 4.9. *If R is right h-weakly regular hemiring and the set of all right h-ideals of R is totally ordered then every right h-ideal of R is an h-prime right h-ideal of R.*

Proof. Let A, B, C be right h-ideals of R such that $AB \subseteq C$. Since the set of all right h-ideals of R is totally ordered, so we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then $A = \overline{AA} \subseteq \overline{AB} \subseteq C$. If $B \subseteq A$ then $B = \overline{BB} \subseteq \overline{AB} \subseteq C$. Thus C is an h-prime right h-ideal. □

Theorem 4.10. *If every fuzzy right h-ideal of a hemiring R is fuzzy h-prime right h-ideal, then R is right h-weakly regular hemiring and the set of fuzzy h-ideals of R is totally ordered.*

Proof. Suppose R is a hemiring in which each fuzzy right h-ideal is fuzzy h-prime right h-ideal. Let λ be a fuzzy right h-ideal of R then $\lambda \odot_h \lambda$ is also a fuzzy right h-ideal of R. As $\lambda \odot_h \lambda \leq \lambda \odot_h \lambda \Rightarrow \lambda \leq \lambda \odot_h \lambda$. But $\lambda \odot_h \lambda \leq \lambda$ always. Hence $\lambda = \lambda \odot_h \lambda$. Thus R is right h-weakly regular hemiring.

Let λ, μ be any fuzzy h-ideals of R. Then $\lambda \odot_h \mu \leq \lambda \wedge \mu$. As $\lambda \wedge \mu$ is a fuzzy right h-ideal of R so a fuzzy h-prime right h-ideal. Thus either $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. That is, either $\lambda \leq \mu$ or $\mu \leq \lambda$. □

Theorem 4.11. *If R is right h-weakly regular hemiring and the set of all fuzzy right h-ideals of R is totally ordered, then every fuzzy right h-ideal of R is a fuzzy h-prime right h-ideal of R.*

Proof. Let λ, μ, ν be fuzzy right h-ideals of R such that $\lambda \odot_h \mu \leq \nu$. Since the set of all fuzzy right h-ideals of R is totally ordered, we have $\lambda \leq \mu$ or $\mu \leq \lambda$. If $\lambda \leq \mu$ then $\lambda = \lambda \odot_h \lambda \leq \lambda \odot_h \mu \leq \nu$. If $\mu \leq \lambda$, then $\mu = \mu \odot_h \mu \leq \lambda \odot_h \mu \leq \nu$. Thus ν is a fuzzy h-prime right h-ideal. $\square$

Example 4.12. Consider the set $R = \{0, x, 1\}$ in which the "sup"($\vee$) and "inf" ($\wedge$) are defined by the chains $0 < 1 < x$ and $0 < x < 1$. On the set R, define $+ = \vee$ and $\cdot = \wedge$. Then $(R, +, \cdot)$ is a hemiring with the following tables:

$+$	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

The crisp right ideals of R are $\{0\}, \{0, x\}, \{0, x, 1\}$. The only right h-ideal of R is $\{0, x, 1\}$, which is idempotent. Obviously R is right h-weakly regular hemiring and $\{0, x, 1\}$ is h-prime and thus h-semiprime.

In order to examine the right fuzzy h-ideals of R, we observe the following facts concerning R.

<u>Fact 1.</u>

Let $\lambda : R \rightarrow [0, 1]$ be a fuzzy subset of R. Then λ is a fuzzy right ideal of R if and only if $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Proof. Suppose $\lambda : R \rightarrow [0, 1]$ be a fuzzy right ideal of R. Since $0 = x \cdot 0 = 1 \cdot 0$ so $\lambda(0) \geq \lambda(x)$ and $\lambda(0) \geq \lambda(1)$. Also $\lambda(x) = \lambda(1 \cdot x) \geq \lambda(1)$. Thus $\lambda(0) \geq \lambda(x) \geq \lambda(1)$.

Conversely, suppose that $\lambda : R \rightarrow [0, 1]$ is a fuzzy subset of R such that $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. By the definition of $+$ defined on R, we have $m + m^{'} = m$ or $m^{'}$ for every $m, m^{'} \in R$, and certainly $\lambda(m) \wedge \lambda\left(m^{'}\right) \leq \lambda(m)$ and $\lambda(m) \wedge \lambda\left(m^{'}\right) \leq \lambda\left(m^{'}\right)$. Thus $\lambda\left(m + m^{'}\right) \geq \lambda(m) \wedge \lambda\left(m^{'}\right)$. By the definition of $\cdot$ defined on R, it is easy to verify that $\lambda(ma) \geq \lambda(m)$ for all m, a in R. Hence λ is a fuzzy right ideal of R. $\square$

<u>Fact 2.</u>

Let $\lambda : R \rightarrow [0, 1]$ be a fuzzy subset of R. Then λ is a fuzzy right h-ideal of R if and only if $\lambda(0) = \lambda(x) = \lambda(1)$.

Proof. Suppose $\lambda : R \rightarrow [0, 1]$ be a fuzzy right h-ideal of R. Then by the Fact 1 $\lambda(0) \geq \lambda(x) \geq \lambda(1)$. Since $1 + 0 + 1 = 0 + 1$, so $\lambda(1) \geq \lambda(0) \wedge \lambda(0) = \lambda(0)$. Thus $\lambda(0) = \lambda(x) = \lambda(1)$.

Conversely, suppose that $\lambda : R \rightarrow [0, 1]$ be a fuzzy subset of R such that $\lambda(0) = \lambda(x) = \lambda(1)$ then by the Fact 1, λ is a fuzzy right ideal of R.

If $x + a + z = b + z$ for $a, b, x, z \in R$ then $\lambda(x) = \lambda(a) \wedge \lambda(b)$. So λ is a fuzzy right h-ideal of R. $\square$

<u>Fact 3.</u>

All fuzzy right h-ideal of R in the above example are idempotent.

Proof. Since each $x \in R$ can be expressed as $x + a_1b_1 + z = a_2b_2 + z$ for some $a_1, b_1, a_2, b_2, z \in R$ and each fuzzy right h-ideal of R is a constant function, so $\lambda \odot_h \lambda = \lambda$ for each fuzzy right h-ideal of R. □

Thus each fuzzy right h-ideal of R is fuzzy h-semiprime. Also each fuzzy right h-ideal of R is fuzzy h-prime. Because $\lambda \odot_h \mu = \lambda \wedge \mu$ and $\lambda \odot_h \mu \leq \nu \Rightarrow \lambda \wedge \mu \leq \nu$. As each fuzzy h-ideal is constant so either $\lambda \wedge \mu = \lambda$ or $\lambda \wedge \mu = \mu$. Thus $\lambda \leq \nu$ or $\mu \leq \nu$.

5. Right pure h-ideals

In this section we define right pure h-ideals of a hemiring R and also right pure fuzzy h-ideals of hemiring R. We prove that every two-sided h-ideal I of a hemiring R is right pure if and only if for every right h-ideal A of R, we have $A \cap I = \overline{AI}$.

Definition 5.1. An h-ideal I of a hemiring R is called right pure if for each $x \in I$, $x \in \overline{xI}$, that is for each $x \in I$ there exist $a, b \in I$ and $z \in R$ such that $x + xa + z = xb + z$.

Example 5.2. Consider the hemiring $R = \{0, a, b\}$ with the following operations

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

The only h-ideal of R is R itself and is right pure.

Lemma 5.3. *An h-ideal I of a hemiring R is right pure if and only if $A \cap I = \overline{AI}$ for every right h-ideal A of R.*

Proof. Suppose that I is a right pure h-ideal of R and A a right h-ideal of R. Then

$$\overline{AI} \subseteq A \cap I.$$

Let $a \in A \cap I$, then $a \in A$ and $a \in I$. Since I is right pure so $a \in \overline{aI} \subseteq \overline{AI}$. Thus $A \cap I \subseteq \overline{AI}$. Hence $A \cap I = \overline{AI}$.

Conversely, assume that $A \cap I = \overline{AI}$ for every right h-ideal A of R. Let $x \in I$. Take A, the principal right h-ideal generated by x, that is, $A = \overline{xR + \mathbb{N}_\circ x}$, where $\mathbb{N}_\circ = \{0, 1, 2, ...\}$. By hypothesis $A \cap I = \overline{AI} = \overline{\overline{(xR + \mathbb{N}_\circ x)}\, I} = \overline{(xR + \mathbb{N}_\circ x)I} \subseteq \overline{xI}$. So $x \in \overline{xI}$. Hence I is a right pure h-ideal of R. □

Definition 5.4. A fuzzy h-ideal λ of a hemiring R is called right pure fuzzy h-ideal of R if and only if $\mu \wedge \lambda = \mu \odot_h \lambda$ for every fuzzy right h-ideal μ of R.

Proposition 5.5. *Let A be a non-empty subset of a hemiring R. Then χ_A, the characteristic function of A, is right pure fuzzy h-ideal of R if and only if A is right pure h-ideal of R.*

Proof. Let A be a right pure h-ideal of R. By Corollary 2.9, χ_A is a fuzzy h-ideal of R.

To prove that χ_A is right pure we have to show that for any fuzzy right h-ideal μ of R, $\mu \wedge \chi_A = \mu \odot_h \chi_A$.

$\mu \odot_h \chi_A \leq \mu \wedge \chi_A$ is always true.

Now if $x \notin A$, then

$$(\mu \wedge \chi_A)(x) = \mu(x) \wedge \chi_A(x) = 0 \leq (\mu \odot_h \chi_A)(x).$$

For the case $x \in A$, as A is right pure h-ideal of R, so there exist $a, b \in A$ and $z \in R$, such that $x + xa + z = xb + z$

As $x, a, b \in A$, this implies $\chi_A(x) = \chi_A(a) = \chi_A(b) = 1$.

Now,

$$\begin{aligned}
(\mu \odot_h \chi_A)(x) &= \sup_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a_j' b_j' + z} \left[\bigwedge_{i=1}^{m} [\mu(a_i) \wedge \chi_A(b_i)] \wedge \bigwedge_{j=1}^{n} \left[\mu(a_j') \wedge \chi_A(b_j')\right] \right] \\
&\geq \bigvee_{x+xa+z=xb+z} \min[\mu(x) \wedge \chi_A(a) \wedge \mu(x) \wedge \chi_A(b)] \\
&\geq \min[\mu(x) \wedge \chi_A(x) \wedge \mu(x) \wedge \chi_A(x)] \\
&\geq \mu(x) \wedge \chi_A(x) \\
&= (\mu \wedge \chi_A)(x).
\end{aligned}$$

So, in both cases $\mu \odot_h \chi_A \geq \mu \wedge \chi_A$.

Thus, $\mu \wedge \chi_A = \mu \odot_h \chi_A$.

So, χ_A is right pure fuzzy h-ideal of R.

Conversely, let χ_A be right pure fuzzy h-ideal of R. Then by Corollary 2.9, A is an h-ideal of R. Let I be a right h-ideal of R, then χ_I is a fuzzy right h-ideal of R. Hence by hypothesis and Proposition 2.13,

$$\chi_{\overline{IA}} = \chi_I \odot_h \chi_A = \chi_I \wedge \chi_A = \chi_{I \cap A}.$$

Thus $\overline{IA} = I \cap A$. So A is right pure h-ideal of R. □

Proposition 5.6. *Let R be a hemiring then the intersection of right pure h-ideals of R is a right pure h-ideal of R.*

Proof. Let A, B be right pure h-ideals of R and I be any right h-ideal of R. Then

$$\begin{aligned}
I \cap (A \cap B) &= (I \cap A) \cap B \\
&= (\overline{IA}) \cap B \quad \text{because } A \text{ is right pure} \\
&= \overline{(\overline{IA})B} \quad \text{because } B \text{ is right pure and } (\overline{IA}) \text{ is a right } h\text{-ideal} \\
&= \overline{(IA)B} \\
&= \overline{I(AB)} = \overline{I(A \cap B)}.
\end{aligned}$$

Hence $A \cap B$ is a right pure h-ideal of R. □

Proposition 5.7. *Let λ_1, λ_2 be right pure fuzzy h-ideals of R, then so is $\lambda_1 \wedge \lambda_2$.*

Proof. Let λ_1 and λ_2 be right pure fuzzy h-ideals of R then $\lambda_1 \wedge \lambda_2$ is a fuzzy h-ideal of R. We have to show that, for any fuzzy right h-ideal μ of R, $\mu \odot_h (\lambda_1 \wedge \lambda_2) = \mu \wedge (\lambda_1 \wedge \lambda_2)$.

Since λ_2 is right pure fuzzy h-ideal of R so it follows that $\lambda_1 \odot_h \lambda_2 = \lambda_1 \wedge \lambda_2$.

Hence

$$\mu \odot_h (\lambda_1 \odot_h \lambda_2) = \mu \odot_h (\lambda_1 \wedge \lambda_2).$$

Also,

$$\begin{aligned} \mu \wedge (\lambda_1 \wedge \lambda_2) &= (\mu \wedge \lambda_1) \wedge \lambda_2 \\ &= (\mu \odot_h \lambda_1) \wedge \lambda_2 && \text{since } \lambda_1 \text{ is right pure} \\ &= (\mu \odot_h \lambda_1) \odot_h \lambda_2 && \text{since } \mu \odot_h \lambda_1 \text{ is a fuzzy right } h\text{-ideal of } R \\ &= \mu \odot_h (\lambda_1 \odot_h \lambda_2). \end{aligned}$$

Thus $\mu \wedge (\lambda_1 \wedge \lambda_2) = \mu \odot_h (\lambda_1 \wedge \lambda_2)$. □

Proposition 5.8. *The following statements are equivalent for a hemiring R with identity :*

1. *R is right h-weakly regular hemiring.*
2. *All right h-ideals of R are h-idempotent (A right h-ideal B of R is h-idempotent if $\overline{B^2} = B$).*
3. *Every h-ideal of R is right pure.*

Proof. (1) $\Leftrightarrow$ (2) By Proposition 3.2.

(1) $\Rightarrow$ (3) Let R be right h-weakly regular hemiring. Let I and A be h-ideal and right h-ideal of R, respectively, then $A \cap I = \overline{AI}$.

Thus by Lemma 5.3, A is right pure.

(3) $\Rightarrow$ (1) Let I be an h-ideal of R and A a right h-ideal of R, then by hypothesis I is right pure and so $A \cap I = \overline{AI}$. Thus by Proposition 3.2, R is right h-weakly regular hemiring. □

Theorem 5.9. *The following statements are equivalent for a hemiring R with identity :*

1. *R is right h-weakly regular hemiring.*
2. *All right h-ideals of R are h-idempotent.*
3. *Every h-ideal of R is right pure.*
4. *Every fuzzy h-ideal of R is right pure.*

If R is assumed to be commutative, then the above assertions are equivalent to

5. *R is h-hemiregular.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Proposition 5.8.

(1) $\Rightarrow$ (4) Let λ and μ be fuzzy right and two sided h-ideals of R, respectively. Then $\lambda \wedge \mu$ is a fuzzy right h-ideal of R. By Theorem 2.14, $\lambda \odot_h \mu \leq \lambda \wedge \mu$. By hypothesis,

$$(\lambda \wedge \mu) = (\lambda \wedge \mu) \odot_h (\lambda \wedge \mu) \leq \lambda \odot_h \mu$$

Hence $\lambda \odot_h \mu = \lambda \wedge \mu$. Thus λ is right pure.

(4) $\Rightarrow$ (1) Let B be a right h-ideal of R and A be a two-sided h-ideal of R then the characteristic functions χ_B and χ_A of B and A are fuzzy right and fuzzy two-sided h-ideal of R, respectively. Hence by hypothesis

$$\chi_B \odot_h \chi_A = \chi_B \wedge \chi_A \implies \chi_{\overline{BA}} = \chi_{B \cap A} \implies \overline{BA} = B \cap A.$$

Thus by Proposition 3.2, R is right h-weakly regular hemiring.

Finally If R is commutative, then by Theorem 2.15, (1) $\Leftrightarrow$ (5). □

Example 5.10. Consider the hemiring $R = \{0, a, b\}$ with the following operations

$+$	0	a	b
0	0	a	b
a	a	b	0
b	b	0	a

$\cdot$	0	a	b
0	0	0	0
a	0	a	b
b	0	b	a

R is a right weakly regular hemiring, so each fuzzy h-ideal of R is right pure.

6. Purely prime h-ideals

In this section we define purely prime h-ideals and purely prime fuzzy h-ideals of a hemiring R and study some basic properties of these ideals.

Definition 6.1. A proper right pure h-ideal I of a hemiring R is called purely prime if for any right pure h-ideals A, B of R, $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

If A, B are right pure h-ideals of R then $A \cap B = \overline{AB}$. Thus the above definition is equivalent to $\overline{AB} \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Definition 6.2. A proper right pure h-ideal μ of a hemiring R is called purely prime if for any right pure fuzzy h-ideals λ, δ of R, $\lambda \wedge \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.

If λ, δ are right pure fuzzy h-ideals of R, then $\lambda \wedge \delta = \lambda \odot_h \delta$. Thus the above definition is equivalent to $\lambda \odot_h \delta \leq \mu \Rightarrow \lambda \leq \mu$ or $\delta \leq \mu$.

Proposition 6.3. *Let R be a right h-weakly regular hemiring with identity and I be an h-ideal of R. Then the following assertions are equivalent:*

(1) For h-ideals A, B of R, $A \cap B = I \Rightarrow A = I$ or $B = I$.
(2) $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$.

Proof. (1) $\Rightarrow$ (2) Suppose A, B are h-ideals of R such that $A \cap B \subseteq I$. Then by Theorem 3.7, $I = \overline{(A \cap B) + I} = \overline{(A + I)} \cap \overline{(B + I)}$. Hence by hypothesis $I = \overline{(A + I)}$ or $I = \overline{(B + I)}$, i.e., $A \subseteq I$ or $B \subseteq I$.

(2) $\Rightarrow$ (1) Suppose A, B are h-ideals of R such that $A \cap B = I$. Then $I \subseteq A$ and $I \subseteq B$. On the other hand by hypothesis $A \subseteq I$ or $B \subseteq I$. Thus $A = I$ or $B = I$. □

Proposition 6.4. *Let R be a right h-weakly regular hemiring. Then any proper right pure h-ideal of R is contained in a purely prime h-ideal of R.*

Proof. Let I be a proper right pure h-ideal of an h-weakly regular hemiring R and $a \in R$ such that $a \notin I$. Consider the set

$$X = \{J_p : J_p \text{ is a proper right pure } h\text{-ideal of } R \text{ such that } I \subseteq J_p \text{ and } a \notin J_p\}.$$

Then $X \neq \phi$ because $I \in X$. By Zorn's Lemma this family contains a maximal element, say M. This maximal element is purely prime. Indeed, let $A \cap B = M$ for some right pure h-ideals A, B of R. If A, B both properly contains M, then by the maximality of M, $a \in A$ and $a \in B$. Thus $a \in A \cap B = M$, which is a contradiction. Hence either $A = M$ or $B = M$. □

Proposition 6.5. *Let R be a right h-weakly regular hemiring. Then each proper right pure h-ideal is the intersection of all purely prime h-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 4.3. □

Proposition 6.6. *Let R be a right h-weakly regular hemiring. If λ is a right pure fuzzy h-ideal of R with $\lambda(a) = t$ where $a \in R$ and $t \in [0, 1]$, then there exists a purely prime fuzzy h-ideal μ of R such that $\lambda \leq \mu$ and $\mu(a) = t$.*

Proof. The proof is similar to the proof of Proposition 4.4. □

Theorem 6.7. *Let R be a right h-weakly regular hemiring. Then each proper fuzzy right pure h-ideal is the intersection of all purely prime fuzzy h-ideals of R which contain it.*

Proof. Proof is similar to the proof of Theorem 4.5. □

References

[1] A. W. Aho and J. D. Ullman, Introduction to Automata Theory, Languages and Computation, Addison Wesley, Reading MA, 1976.
[2] J. Ahsan, K. Saifullah and M. Farid Khan, Fuzzy semirings, Fuzzy Sets and Systems 60 (1993) 309–320.
[3] G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colleq. Publications, 1954.
[4] W. A. Dudek, M. Shabir and R. Anjum, Characterizations of hemirings by their h-ideals, Comput. Math. Appl. 59 (2010) 3167–3179.
[5] S. Eilenberg, Automata, Languages and Machines, Acedmic press, New York, 1974.
[6] S. Ghosh, Fuzzy k-ideals of semirings, Fuzzy Sets and Systems 95 (1998) 103–108.
[7] K. Glazek, A Guide to Literature on Semirings and their Applications in Mathematics and Information Sciences with Complete Bibliography, Kluwer Acad. Publ. Dodrecht, 2002.
[8] M. Henriksen, Ideals in semirings with commutative addition, Amer. Math. Soc. Notices 6 (1958) 321.
[9] K. Iizuka, On the Jacobson radial of a semiring, Tohoku Math. J. 11 (1959) 409–421.
[10] Y. B. Jun, M. A. Öztürk and S. Z. Song, On fuzzy h-ideals in hemiring, Inform. Sci. 162 (2004) 211–226.
[11] C. B. Kim and M. Park, k-fuzzy ideals in semirings, Fuzzy Sets and Systems 81 (1996) 281–286.
[12] D. R. LaTorre, On h-ideals and k-ideals in hemirings, Publ. Math. (Debrecen) 12 (1965) 219–226.
[13] E. T. Lee and L. A. Zadeh, Note on fuzzy languages, Inform. Sci. 1 (1969) 421–434.
[14] X. Ma and J. Zhan, Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings, Inform. Sci. 179 (2009) 1249–1268.
[15] X. Ma and J. Zhan, On fuzzy h-ideals of hemirings, J. Syst. Sci. Complex. 20 (2007) 470–478.
[16] C. V. Negoita and D. A. Ralesen, Applications of Fuzzy Sets to System Analysis, Birkhauser, Basel. 1975.
[17] H. S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc. 40 (1934) 914–920.
[18] W. G. Wee and K. S. Fu, A formulation of fuzzy automata and its applications as a model of learning system, IEEE Trans. Systems Sci. Cybernetics SSC-5 (1969) 215–233.
[19] Y. Yin and H. Li, The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform. Sci. 178 (2008) 3451–3464.
[20] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.
[21] J. Zhan, On properties of fuzzy left h-ideals in hemirings, Int. J. Math. Math. Sci. 19 (2005) 3127–3144.
[22] J. Zhan and B. Davvaz, L-fuzzy h-ideals with operators in hemirings, Northeast Math. J. 23 (2007) 1–14.
[23] J. Zhan and W. A. Dudek, Fuzzy h-ideals of hemirings, Inform. Sci. 177 (2007) 876–886.

MUHAMMAD SHABIR (mshabirbhatti@yahoo.co.uk) – Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

RUKHSHANADA ANJUM (rukh_28@hotmail.com) – Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan